Ability Grouping in Contests*

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Abstract

This paper considers a situation in which students with heterogeneous ability types are grouped into different competitions for performance ranking. A planner can group the students and design prize structure in each competition in order to maximize the weighted total performance subject to a no-child-left-behind requirement. We show that, whatever the type-specific weights are, separating – assigning students with the same ability together – is superior to mixing – assigning students with different abilities together. Moreover, we also characterize the associated optimal prize structures.

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1 Introduction

Consider a school that needs to assign a group of students to different classrooms. Should the school group students with similar abilities together – a practice known as "tracking"– so that high ability students are separated from low ability ones, or should the school have mixed classrooms in which students of different abilities are grouped together? Tracking was very common in US schools but became less popular in the late 1980s due to the criticism of trapping students of low socioeconomic status in low-level groups. However, tracking, often referred to as ability grouping, has returned to the attention of educators recently. According to Yee (2013), "*... of the fourth-grade teachers surveyed, 71 percent said they had grouped students by reading ability in 2009, up from 28 percent in 1998.*" Different grouping policies can be observed not only over time but also across different countries. For instance, in Germany, pupils after primary schooling are grouped into three types of secondary schools to receive training for blue-collar apprenticeships, apprenticeship training in white-collar occupations, or training for further education. In contrast, tracking was explicitly discontinued in China in 2006.[1] Not surprisingly, tracking has also been a controversial topic in the economic literature on education, and there has been a long debate on this issue from many different perspectives: students' achievement, equity, and even morality.[2]

Lazear (2001) studies tracking when students are awarded according to their absolute performance. He shows that tracking results in higher total performance than mixing does. In many competitions, students' relative performance ranking is also important. For example, college admission in many countries is based on percentile of students' exam scores.[3] In this paper, we shut down the effect of absolute performance and study the competition based on relative performance. To our knowledge, this is the first paper revisiting the question of ability grouping in the new context of contests, in which students are awarded according to their relative performance ranking.

Specifically, this paper investigates the competitive effects of tracking and asks whether it enhances students' performance when their grades or rewards depend on their relative performance.[4] Two features are important to a social planner, e.g., a school.

First, non-performance is not desirable. For example, the No Child Left Behind Act of 2001 requires schools in US to guarantee that nearly all students meet some minimum

[1] Policies that forbade tracking in schools started in the 1990s, and a national law was passed in 2006. In contrast, tracking remains common in Chinese universities.

[2] See, for instance, Loveless (2013).

[3] For example, college admission in Australia is based on Australian Tertiary Admission Rank, which is a percentile rank.

[4] We do not consider the equity issue nor the effects of tracking on the quality of instruction. If students are tracked, the classes are more homogeneous and therefore they could be easier to teach. See, for instance, Duflo et al. (2011).

skill levels in reading, writing, and arithmetic.

Second, the planner may weigh students of different abilities differently. For example, a school may care more about high ability students' performance than others. As a result, it is important to examine how different weights affect the comparison of tracking and mixing. There are empirical evidences showing that ability grouping may benefit students with higher abilities but hurts those with lower abilities (e.g. Ding and Lehrer 2007). Then, if a planner attaches more weight to the lower-ability students, it seems optimal to mix students of different ability levels. Our results show otherwise.

In this paper, we apply insights from contest theory to analyze ability grouping. Our model builds on Siegel's (2010) model of all-pay contests by introducing heterogeneous prizes, performance spillovers and a planner who can allocate students and prize money across contests. Specifically, suppose that there is a fixed amount of prize money and a group of students with different ability types. The students of the same type have the same constant marginal cost of performance/effort. We do not distinguish effort and performance in this paper.[5] A lower cost of performance represents a higher ability. A planner can assign the students into any number of contests and divide the money as potentially heterogeneous prizes in the contests.

In each contest, the students choose their performance simultaneously. The player with the highest performance receives the highest prize, the player with the second highest performance receives the second prize, and so on. There may be performance spillovers within a contest. That is, a player may benefit from other students' performance in the contest. The no-child-left-behind requirement imposes a restriction on the planner, so the planner has to ensure that as an equilibrium outcome, no student chooses non-performance with positive probability. The planner attaches type-specific weights to the player's performance, and wants to maximize the weighted total performance subject to the no-child-left-behind requirement. Our main result is:

Proposition 1. *Given any weights and the no-child-left-behind requirement, the weighted total expected performance is maximized only if the students are separated.*

Therefore, with a well-designed award system, tracking/separating – assigning students with the same ability together – is always superior to mixing – assigning students with different abilities together. Moreover, this result is independent of how the planner weighs different ability groups. Specifically, with a well-designed award system, separating is always optimal, even if the school cares little about the high ability students.

In contest theory, it is a well-known result that if participants' abilities are symmetric

[5] In other words, we assume a deterministic relation between effort and performance such that one unit of effort leads to one unit of performance.

and commonly known, intense competition drives the participants' payoffs to zero (e.g., Hillman and Riley 1989). This paper applies the insight to the context of ability grouping, and argues that by separating the students according to their abilities, we can introduce intense competition among asymmetric players. In addition, we generalize the insight to accommodate the new features arising from the environments of ability grouping, such as spillovers, the no-child-left-behind requirement, and heterogeneous weights for students. We also show that that such intense competition cannot be implemented in contests with mixed student types if there are spillovers or the no-child-left-behind requirement. As a result, for any set of contests with mixed students, there always exist contests of separated students with at least as much total expected performance for each ability type and a higher total expected performance for at least one ability type. In other words, having separated students dominates having mixed students in terms of the total expected performance of different types.

It is important that the planner can choose prize structures while grouping the students. If the planner cannot choose the prize structures, Xiao (2016) shows, in an example, that separating may result in higher or lower total expected performance than mixing. We also characterize the optimal prize structures.

There are several challenges that we have to overcome in order to establish these results. First, with asymmetric players and heterogeneous prizes, the equilibrium of an all-pay contest may involve complicated mixed strategies (e.g., Bulow and Levin 2006, González-Díaz and Siegel 2013 and Olszewski and Siegel 2016). To our knowledge, equilibrium characterization for general asymmetric contests with heterogeneous prizes is still an open question. The techniques in this paper allows us to circumvent the difficulty and characterize the optimal way to group players.

Second, there could be multiple equilibria. Given a particular prize structure, equilibrium selection may change the comparison between separating and mixing (e.g., Xiao 2016). The results in this paper apply to all equilibria, and therefore are robust to equilibrium selection.

Finally, the generality of the model also imposes extra challenges. The current paper does not restrict the number of contests, the prize structures, or the player composition in contests, which means the planner has to compare a large number of choices. Moreover, this paper accommodates a very general objective function for the planner: she can assign asymmetric type-specific weights to the players' performance, and she can impose different minimal performance requirements on different types.

Literature The works on affirmative action discuss separating students according to their race, gender, nationality, etc. For example, Fu (2006) uses a contest model to study student competition and shows that, to maximize its incoming students' weighted total scores, a college may adopt an admission rule that favors the minority. Such a policy can be viewed as a handicapping rule in the same contest. In this paper we adopt the same objective to maximize the weighted total performance, but we study a different question. We study how to allocate students of different abilities across different contests. Bodoh-Creed and Hickman (2018) study different affirmative action policies on college admissions, where students have private information on their performance costs. One of the policies utilizes a quota system, in which the two groups compete separately for their respective reserved seats. The competition effort in their model is wasteful except signaling the students' ability. In contrast, the effort is not wasteful in this paper, and the social planner wants to maximize students' learning effort. Similar to our paper, Olszewski and Siegel (2019) use all-pay contests to study competition among students. They show that the optimal performance-disclosure policies involves pooling, which assigns the same score to a subset of students with different performances. Their paper focuses on policies in a single contest, while we focus on policies that divide students into different contests to compete.

Studies on peer effects also discuss grouping players across competitions (e.g., Board 2009 and Fruehwirth 2013). The players' payoffs in these papers depend on their absolute performance, while we consider a different situation in which students are awarded according to their relative performance, or the ranking of their performance.

There is a big literature on contests, and this paper's main difference from the literature lies in the new features arising from the environments of ability grouping, such as heterogeneous weights to different ability types and the no-child-left-behind requirement. Among them, some study player allocations with a fixed prize structure. In contrast, this paper considers the joint decision on prize structures and student allocations. For example, in an all-pay auction model (with a single winner), Baye et al. (1993) show that a politician may extract higher total rent by excluding the lobbyists, who have higher valuations of winning, from the competition. Parreiras and Rubinchik (2020) study allocations of ex ante asymmetric players across two all-pay auctions. They find that if the players are similar, the total revenue is maximized by two equal auctions separating the players of higher costs from those of lower costs.

The following papers study the effect of merging prizes and players in small contests into a grand contest. For example, Moldovanu and Sela (2006) compare different ways to group ex ante symmetric players across contests. They find that total expected effort is maximized by a grand contest including all players, while the expected highest effort is maximized by splitting players into multiple contests and letting winners of each contest compete in a final contest. Fu and Lu (2009) find that merging multi-winner contests of symmetric players can increase total expected effort. In contrast, this paper considers players with asymmetric abilities. The asymmetry is crucial as we are considering whether players should be separated according to their abilities.

The literature on status competition anaylizes the optimal way to divide players into different status categories when the players' payoffs depend directly on their status (see e.g., Moldovanu et al. 2007 and Dubey and Geanakoplos 2010). In our paper, performance ranking does not affect players' payoffs directly. Instead, the payoffs depend on the prizes awarded according to the players' ranking.

The remainder of the paper is organized as follows. Section 2 introduces the model without spillovers. Section 3 presents the main results on optimal grouping and optimal prize structures. Section 4 generalizes the results to the case with spillovers. Section 5 discusses several extensions.

2 Model

There is one unit of prize money and a finite set N of students, or players. The players are of $T \geq 2$ different types, and there are n_t players of each type $t \in \{1, 2, ..., T\}$. We assume $n_t \geq 2$ for all t, so there are similar players of each type.[6] Players of t-type have the same marginal cost of performance/effort, c_t. We do not distinguish between effort and performance. Without loss of generality, we assume $0 < c_1 < c_2 < ... < c_T$.

The decision of a school, or a planner, has two parts: assigning the players into any number of contests, and dividing the prize money as prizes for each of the contests. Specifically, the planner's choices can be represented by a partition of the players $\mathcal{P}$ and a prize structure $\mathcal{V}$.[7] The partition $\mathcal{P}$ is a family of non-empty subsets of N such that N is a disjoint union of the subsets. Suppose partition $\mathcal{P}$ consists of M sets, then we have $\mathcal{P} = \{P^1, ..., P^M\}$. The prize structure $\mathcal{V}$ is a family of vectors $\{\mathbf{v}^1, ..., \mathbf{v}^M\}$, where for any $m = 1, ..., M$, the vector $\mathbf{v}^m \in [0, 1]^{|P^m|}$ and $|P^m|$ is the number of players in the set P^m. A zero entry of $\mathbf{v}^m$ means one of the prizes is zero. Therefore, the partition $\mathcal{P}$ and prize structure $\mathcal{V}$ characterize the M contests. In particular, the subset $P^m \subset N$ is the set of players assigned to contest m, and $\mathbf{v}^m$ represents the prizes in contest m. Throughout the paper, the superscripts are indexes of contests. Notice that there is no restriction on the number of contests, so the planner may assign all the players into one

[6] This assumption is relaxed in Section 5.2.

[7] Allowing the planner to choose prizes does not preclude the possibility that players' values of ranking are partly determined outside the tournaments. For instance, if the planner wants to have a first prize of \$200 thousand and she knows that the champion will also receive an endorsement deal worth \$100 thousand, then she can simply award the difference of \$100 thousand to the champion. A similar argument applies when a school has a budget for scholarships.

contest, that is, $\mathcal{P} = \{N\}$.

Next, we describe the competition in each contest. In a contest characterized by its player set P^m and prize vector $\mathbf{v}^m$, all the players in P^m choose their performance/effort levels in $[0, +\infty)$ simultaneously. The prizes are monetary, so a prize has the same value to all players. However, players with different performance rankings may receive prizes of different monetary values. Specifically, the player with the highest performance receives the highest prize in $\mathbf{v}^m$; the player with the second highest performance receives the second highest prize in $\mathbf{v}^m$; and so on. In the case of a tie, the prizes are allocated randomly such that no tying player loses with certainty.[8] If a player wins a prize, his payoff is his prize net of his cost of performance. If a player wins no prize, his payoff is zero minus his cost of performance. All players are risk neutral. A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players.

The planner's objective has three parts. First, the planner attaches weight α_t to a t-type player's performance, and she wants to maximize the weighted total expected performance. We assume that $\alpha_t \in (0, 1)$ and $\sum_{t=1}^{T} \alpha_t = 1$. The weight α_t represents the relative importance of t-type players' performance to the planner. Let $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_T)$ be the vector of weights. Second, the planner has a no-child-left-behind requirement. Let s_{i_t} be the performance of player i_t of t-type. Then, the planner wants to ensure $s_{i_t} > 0$ almost surely (a.s.) for every player $i_t \in N$. Note that the requirement is on the planner's objective instead of players' choice. This means that the planner has to provide well-designed incentives so that the players do not choose non-performance even if they could. Third, the planner has a no-extreme-prize-allocation requirement: the total expected prize won by t-type players is at least $\underline{W}_t > 0$.[9] This requirement may represent the concern that students with socioeconomic advantages end up winning too many scholarships.[10] Because s_{i_t} is restricted to an open set, this requirement ensures the existence of an optimal prize structure.[11] In addition, we assume $\sum_{t=1}^{T} \underline{W}_t \leq 1$, otherwise the requirement can never be satisfied.

We introduce some definitions below that are important in later analysis. We say a partition and prize structure pair $(\mathcal{P}, \mathcal{V})$ is feasible if the total prize in $\mathcal{V}$ is no more than 1. Denote by Φ as the set of all feasible pairs of partition and prize structure. Given

[8] In many tournaments (for example, in golf), ties are resolved by sharing the prizes. As an example, if two players tie with the second-highest performance, then each receives the average of the second and third prize. Our formulation allows this kind of sharing.

[9] Our analysis applies if we impose a more restrictive version of no-extreme-prize-allocation requirement: the total prize won by t-type players is at least $\underline{W}_t$ almost surely.

[10] See, e.g., Saul (2012).

[11] Remark 2 discusses the consequence of removing the no-child-left-behind or no-extreme-prize-allocation restriction.

any feasible $(\mathcal{P}, \mathcal{V})$, let $\mathbb{E}[s_{i_t}]$ be the expected performance of player i_t of t-type in an equilibrium. Define a correspondence $\Pi : \Phi \to 2^{\mathbb{R}_+}$ such that

$$\Pi(\mathcal{P}, \mathcal{V}) = \left\{ \sum_{t=1}^{T} \left(\alpha_t \sum_{i_t=1}^{n_t} \mathbb{E}[s_{i_t}] \right) \right\}$$

where $2^{\mathbb{R}_+}$ is the power set of $\mathbb{R}_+$. Note that $\Pi(\mathcal{P}, \mathcal{V})$ may contain multiple values if there are multiple equilibria. We say $(\mathcal{P}^*, \mathcal{V}^*)$ maximizes $\Pi(\mathcal{P}, \mathcal{V})$ if $\inf \Pi(\mathcal{P}^*, \mathcal{V}^*) \geq \sup \Pi(\mathcal{P}, \mathcal{V})$ for any other feasible $(\mathcal{P}, \mathcal{V})$, where the infimum and supremum are taken over the multiple values in $\Pi(\mathcal{P}^*, \mathcal{V}^*)$ or $\Pi(\mathcal{P}, \mathcal{V})$.

The planner chooses partition $\mathcal{P}$ and prize structure $\mathcal{V}$ to maximize the weighted total expected performance subject to the no-child-left-behind and no-extreme-prize-allocation requirements. Therefore, her problem is

$$\max_{(\mathcal{P},\mathcal{V})\in\Phi} \Pi(\mathcal{P}, \mathcal{V}) \tag{1}$$

$$\text{s.t.} \quad s_{i_t} > 0 \text{ a.s. for every } i_t \in N \text{ in any equilibrium} \tag{2}$$

$$W_t \geq \underline{W}_t \text{ for every } t \text{ in any equilibrium} \tag{3}$$

where W_t is the total expected winnings of the t-type players. Any violation of restriction (2) or (3) is unacceptable to the planner. We say $(\mathcal{P}^*, \mathcal{V}^*)$ is optimal if $(\mathcal{P}^*, \mathcal{V}^*)$ maximizes $\Pi(\mathcal{P}, \mathcal{V})$ subject to (2) and (3).

3 Optimal Grouping

If a contest has only one player, his equilibrium performance level must be zero, which violates (2). Therefore, we only need to consider contests with at least two players. The planner is said to *separate* the players if each contest contains players of the same type. Otherwise, we say the planner *mixes* the players. In order to separate the players, the planner cannot have fewer than T contests, but she can have more than T by splitting a larger contest into smaller ones. The main result of this paper is as follows.

Proposition 1 *Given any weights and the no-child-left-behind requirement, the weighted total expected performance is maximized only if the players are separated.*

In the remainder of this section, we first prove Proposition 1 through a sequence of lemmas, then specify the associated optimal prize structures in Proposition 2. The lemma below ensures equilibrium existence.

Lemma 1 *In each contest, there exists no Nash equilibrium in pure strategies, but there exists a Nash equilibrium in mixed strategies.*

Siegel (2010) establishes equilibrium existence for contests with homogeneous prizes, and his proof is readily adapted to prove the above lemma, so we omit its proof. Because of the mixed strategies in an equilibrium, we always discuss a player's expected payoff instead of his payoff. Therefore, we simply say "payoff" below instead of "expected payoff".

One of the challenges is that there may be multiple equilibria in a contest. Our method overcomes this challenge by relying on properties true for all equilibria. Lemmas 2-5 below present four such properties. Lemma 2 characterizes a property of atoms in one's mixed strategy, then we use this property to prove Lemma 3.

Lemma 2 *Suppose a player has an atom at performance level s in an equilibrium, that is, he chooses s with strictly positive probability. Then, he receives the lowest prize in the contest with certainty by choosing this performance level.*

The proof is in the appendix. Before discussing the next equilibrium property, we need to introduce some notation. Consider a contest with a player set P^m and a prize vector $\mathbf{v}^m$. We use a cumulative distribution function $G_i^m : [0, +\infty) \rightarrow [0, 1]$ to represent player i's strategy in an equilibrium. If G_i^m assigns probability 1 to a single performance level s_i, it represents a pure strategy s_i. A strategy's support is the smallest closed set that receives probability 1 according to the strategy. Let $\bar{s}_i^m$ be the highest performance in the support of G_i^m. In the contest, given other players' strategies $\mathbf{G}_{-i}^m \equiv (G_j^m)_{j \in P^m \setminus \{i\}}$, player i's expected value of prizes by choosing s_i is denoted as $W(\mathbf{G}_{-i}^m(s_i), \mathbf{v}^m)$, which is also referred to as his expected winnings. The lemma below provides a property of the expected winnings that is useful to prove our main result.

Lemma 3 *Consider a contest with a player set P^m and a prize vector $\mathbf{v}^m$, in which not all the prizes are identical. Then, in any equilibrium,*

$$W(\mathbf{G}_{-i}^m(\bar{s}_j^m), \mathbf{v}^m) \geq W(\mathbf{G}_{-j}^m(\bar{s}_j^m), \mathbf{v}^m) \tag{4}$$

Proof. There are two possibilities. First, suppose $\bar{s}_j^m = 0$. It means player j chooses non-performance with probability 1, then Lemma 2 implies that player j receives the lowest prize with certainty. Hence, $W(\mathbf{G}_{-i}^m(\bar{s}_j^m), \mathbf{v}^m)$ cannot be lower than j's expected winnings in the equilibrium. That is, inequality (4) holds.

Second, suppose $\bar{s}_j^m > 0$. Lemma 2 implies that no player chooses $\bar{s}_j^m$ with positive probability in the equilibrium. Therefore, if player i chooses performance $\bar{s}_j^m$, his performance is higher than j's with certainty, i.e., by choosing $\bar{s}_j^m$ player i beats j for sure.

In addition, they face the same competition from the other players. Hence, player i's expected winnings by choosing $\bar{s}_j^m$ are no less than j's expected winnings by choosing $\bar{s}_j^m$. That is, inequality (4) holds. ■

Hillman and Riley (1989) show that if players are identical, the competition in an all-pay auction is so intense that each player receives zero payoff. The following lemma generalizes the insight to any prize structure. Since a similar result is established by Barut and Kovenock (1998), we omit the proof here.

Lemma 4 *If all the players in a contest are identical, each player's payoff in any equilibrium is the value of the lowest prize. Moreover, if all the prizes except the lowest are positive, there is a unique equilibrium and the equilibrium is symmetric.*

Hillman and Riley also show that if the players are not identical, the highest ability participant may have positive rent. However, this insight may not hold in some contests. Example 1 below illustrates such a contest.

Example 1 *Consider a contest in which four players compete for a prize of value* 1. *The players' marginal costs are* $c_1 = 1, c_2 = 1, c_3 = 2, c_4 = 2$, *so the contest has mixed players but some participants are identical. The contest has a unique equilibrium, in which players* 1 *and* 2 *uses strategies* $G_1(s) = G_2(s) = s$ *and players* 3 *and* 4 *choose pure strategies* $s_3 = s_4 = 0$. *All four players have zero equilibrium payoffs.*

Note that players 3 and 4 choose non-performance, so the no-child-left-behind requirement is violated. This is because the contest has only one prize. As we show in Proposition 2 below, multiple prizes are important for optimal prize structures.

The result below provides a condition under which the property of positive rent holds in contests with any composition of ability types.

Lemma 5 *Consider a contest in which not all players have the same type and not all prizes have the same value. Then, in any equilibrium, in which each player's performance is positive a.s., at least one player's payoff is higher than the lowest prize.*

Proof. Suppose players i and j have different costs, with $c_i > c_j$. Since the payoffs cannot be lower than the lowest prize, it is sufficient to show that players i and j have different payoffs. Assume otherwise that $u_i = u_j$, where u_i and u_j are i and j's payoffs in an equilibrium. In the equilibrium, if every player's performance is positive a.s., then the highest performance in the support of i's strategy is positive, i.e., $\bar{s}_i > 0$. According to Lemma 3, player j's expected winnings at $\bar{s}_i$ are no lower than i's at $\bar{s}_i$. Therefore, if player j deviates to $\bar{s}_i > 0$, his expected winnings are no lower than i's, but his cost is lower than i's. Hence, the deviation results in a payoff of j that is higher than u_i. This is a contradiction because i and j have the same payoff by assumption. ■

Next, we proceed to prove Proposition 1. Given any equilibrium, denote by $S_t \equiv \sum_{i=1}^{n_t} \mathbb{E}[s_{i_t}]$ as the total expected equilibrium performance of all t-type players. Intuitively, for any $(\mathcal{P}, \mathcal{V})$ that mixes players and satisfies (2) and (3), we first introduce another pair $(\mathcal{P}', \mathcal{V}')$ that separates players and satisfies (2) and (3). Then, we verify that the performance outcome $(S_1, ..., S_T)$ with mixed players is *dominated* by the outcome $(S'_1, ..., S'_T)$ with separated players in the sense that $S_t \leq S'_t$ for all t and $S_t < S'_t$ for some t. This shows that separating is superior to mixing for any type-specific weights.

Proof of Proposition 1. Consider a partition and prize structure pair $(\mathcal{P}, \mathcal{V})$ satisfying (2) and (3), where the players are mixed. Take an equilibrium in each contest. Since restriction (2) is satisfied, we must have $\mathbb{E}[s_{i_t}] > 0$ for any i_t in the equilibrium. Given the equilibria, let $(S_1, ..., S_T)$ be the performance outcome, U_t be the t-type players' total payoff, and W_t be their total expected winnings. Requirement (3) implies $W_t \geq \underline{W}_t$. For each player, his payoff equals his expected winnings minus the cost of his expected performance, so we have $U_t = W_t - c_t S_t$, or

$$S_t = (W_t - U_t)/c_t \tag{5}$$

Recall that (2) is satisfied, so in each contest, not all prizes are identical. In addition, note that U_t is the total payoff when the players are mixed, so Lemma 5 implies that $U_t > 0$ for some t.

Consider another pair $(\mathcal{P}', \mathcal{V}')$ such that $\mathcal{P}'$ separates the players into T contests, so each contest contains all the players of a particular type. In addition, suppose $\mathcal{V}'$ assigns prizes of total value W_t to the contests with t-type players and all the prizes except the lowest are positive. Notice that the total winnings of t-type players remains the same at W_t, so $(\mathcal{P}', \mathcal{V}')$ satisfies requirement (3). In addition, since all the players in a contest are identical and all the prizes except the lowest are positive, Lemma 4 implies that the contest has a unique equilibrium and the equilibrium is symmetric. Thus, $(\mathcal{P}', \mathcal{V}')$ also satisfies requirement (2).

Recall that in $\mathcal{V}'$, the lowest prize is zero in every contest, so Lemma 4 implies that all players have zero payoffs. Similar to (5), in the unique equilibrium, the total expected performance of t-type players is $S'_t = (W_t - 0)/c_t$. Comparing to (5), we obtain $S'_t \geq S_t$ for all t. Recall that $U_t > 0$ for some t, so $S'_t > S_t$ for some t. Hence, the performance outcome $(S'_1, ..., S'_T)$ dominates $(S_1, S_2, ..., S_T)$.

Hence, for any prize structure and any partition $(\mathcal{P}, \mathcal{V})$ that satisfy requirement (2) and (3) and mix players, we find another partition and prize structure $(\mathcal{P}', \mathcal{V}')$ such that requirement (2) and (3) are satisfied, the outcome is unique and the outcome dominates that associated with $(\mathcal{P}, \mathcal{V})$. As a result, the weighted total expected performance is never maximized if the players are mixed. ■

Remark 1 *Proposition 1 is robust to small idiosyncratic shocks in the costs. Specifically, suppose player i_t of t-type has a marginal cost $c_t + \varepsilon_{i_t} > 0$, where ε_{i_t} represents the idiosyncratic shock and is commonly known. Then, there exists an $\varepsilon > 0$ such that Proposition 1 remains true if $\max_{i_t \in N} |\varepsilon_{i_t}| < \varepsilon$. See Proposition 7 in the appendix.*

Remark 2 *If the planner does not have the no-child-left-behind requirement $s_{i_t} > 0$ a.s., then separating becomes weakly better than mixing. Specifically, the maximum weighted total expected performance can be achieved if the players are separated, and it may also be achieved if players are mixed. See Example 2. We can also obtain Proposition 1 if we replace the no-child-left-behind requirement by an assumption that each contest contains distinct prizes, which means that the planner has to award higher performance with a higher prize.*

According to Proposition 1, it is never optimal to assign only one player to a contest or to have a contest containing all the players. It is also worth mentioning that there is more than one way to separate the players. For instance, suppose there are two H-type players and four L-type players. The planner can separate the players in two ways. She can have two contests with all the H-type players in one and all the L-type players in the other. Alternatively, she can have one contest with all the H-type players and two other identical contests, each with two L-type players. Proposition 2 below implies that the optimal performance outcome remains the same across the different ways of separating.

Now let us consider the optimal prize structure. As a result of Proposition 1, we only need to find the optimal prizes for separated players. According to Lemma 4, if we transfer value from the lowest prize to the higher prizes in a contest, the total payoff in the contest decreases, therefore the total expected performance increases because of (5). Hence, the optimal lowest prize should be zero in every contest, then all players should have zero payoffs. Therefore, equation (5) implies that the total expected performance of t-type players is V_t/c_t, where V_t is the total value of prizes in the contests containing t-type players. Note that, if the players are separated, the distribution of prize money within a contest has no effect on the total expected performance in the contest as long as all the prizes except the lowest remain positive.[12] As a result, the planner's problem

[12] In different setups where the participants' costs are private information, allocation of prizes would affect the equilibrium performance. See, for example, Moldovanu et al. (2007) and Liu et al. (2018).

(1) becomes a linear programming problem

$$\begin{aligned}\max_{V_1,..,V_T} \quad & \sum_t (\alpha_t V_t / c_t) \\ \text{s.t.} \quad & \sum_t V_t = 1 \\ & V_t \geq \underline{W}_t \text{ for every } t\end{aligned} \tag{6}$$

If $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, it is optimal to minimize V_t, so $V_t = \underline{W}_t$. Based on the analysis above, the proposition below characterizes the optimal prize structures that solve the planner's problem.

Proposition 2 *A prize structure is optimal for separated players if and only if both of the following conditions hold: i) all the prizes except the lowest are positive in each contest, ii) the total value of prizes in the contests of t-type players is $V_t = \underline{W}_t$ if $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$.*

Remark 3 *A single prize, as in an all-pay auction, is not optimal if a contest contains more than two players. For instance, if there are three H-type players and two L-type players, the only way to separate them is grouping all the H-type players in one contest and both L-type players in another. Then, if we assign a single prize in the contest with three H-type players, there exists an equilibrium in which a H-type player chooses non-performance. Therefore, the no-child-left-behind requirement is violated.*

Recall that a performance outcome $(S_1, ..., S_T)$ specifies the total expected performance for types $t = 1, ..., T$. There may be multiple optimal prize structures, but they all result in the same performance outcome. Moreover, according the proof of Proposition 1, given an optimal prize structure and separated players, there is a unique equilibrium in each contest. Therefore, there is a unique performance outcome associated with all optimal prize structures.

So far, we have demonstrated that for separating to be optimal, it should be accompanied with associated optimal prize structures. The share of the budget that is used to motivate the players of t-type is weakly increasing in the weight α_t. If the planner cannot choose the prizes freely, mixing could actually be better than separating. See, for example, Proposition 10 of Xiao (2016).

If the planner does not have the restriction of $s_{i_t} > 0$ almost surely, separating becomes weakly better than mixing, which is illustrated in Example 2.

Example 2 *Suppose there are two players with c_1 and two with c_2, where $c_1 < c_2$. We also assume $\alpha_1 = \alpha_2$, so the planner wants to maximize the total expected performance.*

If we put the two players with c_1 in one contest and the other two in another contest, and if we allocate all the budget to a single prize in the contest with marginal cost c_1, the weighted total expected performance is maximized. Moreover, both players with c_2 choose non-performance. However, the total expected performance can also be maximized by mixing the players. If we put all the players in one contest and allocate the budget to a single prize, the two players with c_1 compete for the prize, and the other two choose non-performance. Therefore, even if the players are assigned to the same contest, they compete as if they are separated. As a result, the total expected performance is also maximized if the players are mixed.

4 Performance Spillovers

Spillovers are generally difficult to analyze, especially for asymmetric players.[13] Here, we follow the literature of aggregate games to introduce additively separable spillovers. More precisely, consider a contest with a set of players P^m and a vector of prizes $\mathbf{v}^m$. Let $\mathbf{s}^m = (s_j^m)_{j \in P^m}$ be a vector of performance levels. If a player i's performance s_i^m is the kth highest in $\mathbf{s}^m$, his payoff is $u_i(\mathbf{s}^m) = v_k^m - (c_i s_i^m - \mu_i \sum_{j \neq i} s_j^m)$, where player j's performance reduces player i's cost of performance by $\mu_i s_j^m$. Notice that a special case is $u_i(\mathbf{s}^m) = v_k^m - c_i(s_i^m - \bar{s}^m)$, where $\bar{s}^m$ is the average performance. This special case is an aggregate game with linear structure. Linear models with aggregate performance are widely used in empirical studies of spillovers in innovation (e.g. Audretsch and Feldman 1996), workplaces (e.g. Mas and Moretti 2009) and education (e.g. Angrist 2014). Acemoglu and Jensen (2013) provide a theoretic study on spillovers through the average or aggregate action in more general competitions.

We assume $\mu_i \geq 0$ for all i, so a player's higher performance does not make another's performance more costly. In addition, we assume μ_i is type-specific. That is, if a player i is of t-type, then $\mu_i = \mu_t$. We refer to $\boldsymbol{\mu} = (\mu_1, ..., \mu_T)$ as the levels of performance spillovers. The parameter μ_t measures how strongly the other players' performance affect a t-type player, and it may be different across types. For example, if $\mu_L > \mu_H = 0$, the L-type players' performance does not affect the H-type players' performance cost, but the H-type players' performance makes the L-type players' performance less costly. If $\mu_t = 0$ for all t, then the setup reduces to that in Section 2.

The following example demonstrates how the spillovers change the equilibrium in a contest.

Example 3 *Consider contest with a prize of value* 1 *and two players with marginal costs* $c_1 = 1$ *and* $c_2 = 2$*. In the case of winning, the payoff is* $u_i = 1 - c_i s_i + \mu s_j$ *for player* i*.*

[13] See Baye et al. (2012) for the case of symmetric players.

The equilibrium strategies are $G_1(s_1) = 2s_1$ for player 1 and $G_2(s_2) = 1/2 + s_2$ for player 2, which are independent of spillover parameter μ. In contrast, the equilibrium payoffs are $u_1 = 1/2 + \mu/8$ for player 1 and $u_2 = \mu/4$ for player 2, which are increasing in μ. This is because one player's payoff may come from others' performance.

It seems convenient that the equilibrium strategies do not change after the introduction of spillovers. However, even in the case of no spillovers, we do not know the equilibrium strategies or the number of equilibria in general. Our approach relies on equilibrium payoffs, but the payoffs differ from the case without spillovers as Example 3 illustrates. Specifically, the properties on equilibrium payoffs in Lemmas 4 and 5 no longer hold. Despite these differences, we generalize below the results in Propositions 1 and 2 to the case with spillovers.

The following lemma is a counterpart of Lemma 1 and establishes equilibrium existence.

Lemma 6 *In each contest with performance spillovers, there exists no Nash equilibrium in pure strategies, but there exists a Nash equilibrium in mixed strategies.*

Proof. Consider a contest with a player set P^m and prize vector $\mathbf{v}^m$. If there are no spillovers, Lemma 1 implies that there is an equilibrium in mixed strategies. Denote the equilibrium as $\mathbf{G}^m$. Given others' strategies in this equilibrium, player i's payoff from choosing s_i is $W(\mathbf{G}^m_{-i}(s_i), \mathbf{v}^m) - c_i s_i$. Consider a different contest with the same players and prizes but with spillovers. In this contest, given others' strategies $\mathbf{G}^m_{-i}$, player i's payoff from choosing s_i becomes $W(\mathbf{G}^m_{-i}(s_i), \mathbf{v}^m) - c_i s_i + \mu_i \sum_{j \neq i} \mathbb{E}[s_j]$, where $\mathbb{E}[s_j]$ is player j's expected performance given his mixed strategy G^m_j. Notice that because of the spillovers, a player's payoff increases by $\mu_i \sum_{j \neq i} \mathbb{E}[s_j]$, which is independent of his performance. Thus, $\mathbf{G}^m$ is also an equilibrium in the contest with spillovers. Therefore, an equilibrium in the contest without spillovers is also an equilibrium in the contest with spillovers. The converse is also true by the same argument. Hence, fix the player set P^m and prize vector $\mathbf{v}^m$, a strategy profile is an equilibrium in the contest with spillovers if and only if it is an equilibrium in the contest without spillovers. Hence, we establish a *strategic equivalence* between the two contests.

Due to the strategic equivalence, Lemma 1 also ensures equilibrium existence in the presence of spillovers. We prove below that the contest with spillovers has no equilibria in pure strategies. Suppose otherwise that there is an equilibrium in pure strategies. Then, according to the strategic equivalence, the pure strategy equilibrium in the contest with spillovers remains an equilibrium in the contest without spillovers. This contradicts Lemma 1. ■

As a result of the strategic equivalence established above, Lemmas 2-3 remain true in contests with spillovers, otherwise, in a contest without spillovers, there is an equilibrium violating Lemma 2 or 3. For consideration of space, we omit the proofs for the lemmas' generalization.

Because a player can benefit from others' performance in the presence of spillovers, Lemmas 4 and 5 no longer hold. Instead, we obtain Lemmas 7 and 8, which serve similar roles of Lemmas 4 and 5 in the proof of Proposition 1. Consider a contest $(P^m, \mathbf{v}^m)$ with a player set P^m and prize vector $\mathbf{v}^m$. Let $\underline{v}^m$ be the lowest prize in $\mathbf{v}^m$ and $|P^m|$ be the number of players in the contest. Moreover, given any equilibrium in the contest, denote the total expected performance as S^m and the total value of prizes as V^m.

Lemma 7 *Consider a contest $(P^m, \mathbf{v}^m)$ with spillovers. If all the players in the contest are of t-type, then $S^m = (V^m - \underline{v}^m|P^m|)/c_t$ in any equilibrium.*

The omitted proofs in this section are in the appendix. Note that, with spillovers, the equilibrium payoff is $u_t = \underline{v}^m + \mu_t \sum_{j \neq i} \mathbb{E}[s_j]$ in contrast to the payoffs $u_t = \underline{v}^m$ in Lemma 4. However, Lemma 7 shows that $S^m = (V^m - \underline{v}^m|P^m|)/c_t$, which is also implied by Lemma 4, remains true here. The lemma below generalizes Lemma 5 to the case of spillovers.

Lemma 8 *Consider a contest $(P^m, \mathbf{v}^m)$ with spillovers. If not all the players are of the same type and not all the prizes are of the same value, then in any equilibrium, $u_i \geq \underline{v}^m + \mu_i \sum_{j \neq i} \mathbb{E}[s_j]$ for all players, and the inequality is strict for at least one player.*

Notice that if $\mu_i = 0$ for all players, there are no spillovers and the above lemma reduces to Lemma 5. Given the above lemmas, we can present the main result in this section, which generalizes Proposition 1 to the case of spillovers.

Proposition 3 *Given any weights $\boldsymbol{\alpha}$ and any levels of performance spillovers $\boldsymbol{\mu}$, the weighted total expected performance is maximized only if the players are separated.*

Due to the strategic equivalence, Proposition 2 also characterizes the optimal prize structures in the case of spillovers.

5 Extensions

5.1 Minimal Performance Requirements

In the previous sections, if $\underline{W}_t$, the minimum total prize won by t-type players, is close to zero, a t-type player's expected performance could also be close to zero. This section considers an additional objective of the planner to avoid the small expected performance. In particular, we assume $\underline{W}_t = 0$ and replace the no-child-left-behind requirement $s_{i_t} > 0$ a.s. by $\mathbb{E}[s_{i_t}] \geq r_t$ for some $r_t > 0$. That is, a t-type player's expected performance should be at least $r_t > 0$ in any equilibrium. Note that $r_1, ..., r_T$ need not be the same. As a result, the restrictions in (2) and (3) are replaced by $\mathbb{E}[s_{i_t}] \geq r_t$ for $i_t = 1, ..., n_t$ and $t = 1, ..., T$. If $\sum_t n_t r_t / c_t > 1$, the minimal levels for the expected performance can never be achieved, because the total cost of achieving the minimal levels exceeds the total value of the prizes. Therefore, we assume $\sum_t n_t r_t / c_t \leq 1$. The following result extends Proposition 1 to accommodate the minimal requirements of performance.

Proposition 4 *For any given weights and any minimal expected performance levels, the weighted total expected performance is maximized only if the players are separated.*

The proof is similar to that of Proposition 1, so it is omitted. Now we consider the optimal prize structure. Problem in (6) becomes

$$\begin{aligned} \max_{V_1,..,V_T} \quad & \sum_{t=1}^{T} \alpha_t V_t / c_t \\ \text{s.t.} \quad & \sum_{t=1}^{T} V_t = 1 \\ & V_t/(c_t n_t) \geq r_t \text{ for every } t \end{aligned}$$

If $\alpha_t / c_t < \alpha_{t'} / c_{t'}$ for some $t' \neq t$, it is optimal to minimize V_t, so $V_t = c_t n_t r_t$, which is just enough to maintain the minimal performance requirement. In addition, we also need to ensure the minimal performance requirements for each individual player. Recall that Lemma 4 implies each player in a contest of identical players exhibits the same expected performance. Therefore, in order to ensure the minimal performance requirements are met, the per capita prize should be at least $c_t r_t$ in each contest containing t-type players. Based on the analysis above, the proposition below characterizes the optimal prize structures that solve the planner's problem.

Proposition 5 *A prize structure is optimal for separated players if and only if all of the following conditions hold: i) all the prizes except the lowest are positive in each contest, ii) the total value of prizes in all the contests of t-type players is $V_t = c_t n_t r_t$ if $\alpha_t / c_t < \alpha_{t'} / c_{t'}$ for some $t' \neq t$, and iii) the total value of prizes in a contest containing k_t t-type players is at least $c_t k_t r_t$.*

Remark 4 *It is worth mentioning that, given the optimal partition and prize structure, the previous no-child-left-behind requirement is also satisfied.*

5.2 Distinct Abilities

This section relaxes the assumption $n_t \geq 2$ and considers players with distinct marginal costs. In particular, suppose that there are four players with marginal costs $0 < c_1 < c_2 < c_3 < c_4$. Moreover, $\alpha_t = 1/4$ for $t = 1, 2, 3$ and 4, which means the planner wants to maximize the total expected performance. Because it is never optimal to have one player in a contest, there are three ways to split the players into two contests: $(\{1,2\},\{3,4\})$, $(\{1,3\},\{2,4\})$ and $(\{1,4\},\{2,3\})$. Let v_{ij} be the prize in the contest of players $\{i,j\}$. For simpler analysis, we remove the no-child-left-behind and no-extreme-prize-allocation requirements, although the analysis is similar with these requirements. The following proposition characterizes the optimal partition and prize structure.

Proposition 6 *Consider four players with marginal costs $c_4 > c_3 > c_2 > c_1 > 0$.*

If $\frac{c_2+c_3}{c_3^2} > \max\left(\frac{c_1+c_2}{c_2^2}, \frac{c_3+c_4}{c_4^2}\right)$, the optimal partition is $(\{1,4\},\{2,3\})$ and the optimal prize structure satisfies $v_{14} = 0$ and $v_{23} = 1$.

If $\frac{c_2+c_3}{c_3^2} < \max\left(\frac{c_1+c_2}{c_2^2}, \frac{c_3+c_4}{c_4^2}\right)$, the optimal partition is $(\{1,2\},\{3,4\})$ and the optimal prize structure satisfies $v_{12} = 1$ if $\frac{c_1+c_2}{c_2^2} > \frac{c_3+c_4}{c_4^2}$ and $v_{34} = 1$ if $\frac{c_1+c_2}{c_2^2} < \frac{c_3+c_4}{c_4^2}$.

The proof is in the appendix. The proposition implies that players with similar abilities should be grouped together, which may or may not be the same as separating the lower abilities from the higher abilities. To see why they may be the same, consider fixed c_1 and c_4. If c_2 converges to c_1 and c_3 converges to c_4, Proposition 6 implies that the similar players should be grouped together: $(\{1,2\},\{3,4\})$, where the higher ability players are separated from the lower ability ones. In contrast, for fixed c_1 and c_4, if c_2 and c_3 converge to $(c_1 + c_4)/2$, Proposition 6 implies that it is optimal to assign players 2 and 3 into one contest and players 1 and 4 into another. Therefore, separating higher ability players from the lower ability ones is not optimal.

It would be an interesting extension to consider the optimal grouping problem for more than four players whose marginal costs are all different. As in Proposition 6, the optimal grouping would depend on the distribution of costs. Moreover, since each contest inevitably has asymmetric players, it is very important to characterize the equilibria in asymmetric contests with heterogeneous prizes, which to our knowledge is still an open question. Similarly, extending the model to accommodate constraints on the maximum number of contests or on the number of participants in each contest would lead to the same difficulty in equilibrium characterization.

5.3 Heterogeneous Valuations

Our analysis can be generalized to heterogeneous valuation of the same prize. Consider a contest with players $1, ..., n$ and prizes $v_1 \geq ... \geq v_n \geq 0$. Player i is characterized by $(d_i, x_i) \in \mathbb{R}^2_{++}$. His value of the kth prize is $d_i v_k$ for $k = 1, ..., n$, and his marginal performance cost is x_i. Parameters $d_1, ..., d_n$ represent players' heterogeneous values. Consider another contest with n players and the same prizes. In this contest, player i has marginal cost $\hat{c}_i \equiv x_i/d_i$ and his value of the kth prize is v_k. Then, the contest is the same as in Section 2. Given the same prize, the payoff of player with (d_i, x_i) in the first contest is the payoff of player $\hat{c}_i$ in the second contest multiplied by d_i. Therefore, the two contests have the same set of equilibria. Hence, we can replace c_i in Section 2 by x_i/d_i, and the analysis remains the same.

5.4 Alternative Objectives of the Planner

Our analysis applies to other objectives of the planner. Suppose $T = 2$, so there are two ability types. Suppose the planner's objective is

$$\max_{(\mathcal{P},\mathcal{V})\in\Phi} (S_1 + S_2) - \beta|S_1/n_1 - S_2/n_2| \tag{7}$$

subject to (2) and (3), where $\beta \in (0, 1)$. This means that the planner wants to maximize the total expected performance and she also wants to minimize the difference in the average performance across groups. Because $\beta < 1$, maximizing total performance is more important than minimizing the performance gap. Chan and Eyster (2003) study a similar objective in a study of college admission policies. We can verify that (7) subject to (2) and (3) has the same optimal partition and prize structure as either $\max(1 - \beta/n_1)S_1 + (1 + \beta/n_2)S_2$ subject to (2) and (3) or $\max(1 + \beta/n_1)S_1 + (1 - \beta/n_2)S_2$ subject to (2) and (3). In either case, we transform (7) into a problem as in (1), so the analysis in Section 2 applies.

5.5 Application to Sports

Besides education, the results in this paper are also applicable to a variety of competitions. For instance, in the early nineteenth century, there were no weight classes in boxing. Then eight weight classes were introduced before the Second World War, and nine more were introduced afterwards. The history of other sports such as weightlifting and wrestling also shares similar trend. The heavier athletes have an obvious advantage in strength, so why would we want to group players with similar abilities into the same class and let them compete only within their class? This takes fairness into consideration, and our results suggest that separating athletes according to their abilities could also increase their effort and therefore make matches more entertaining.

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Appendix

Proof of Lemma 2. We first claim that, if two or more players have an atom at performance level s in an equilibrium, all the players who have an atom at s lose with certainty.[14] Let us prove it by contradiction. Suppose that two players, i and j, have an atom at performance level s in an equilibrium, and suppose that player i wins a prize with positive probability by choosing s. Since the tie is broken in such a way that everyone involved wins with positive probability, player j also wins a prize with positive probability by choosing s. In addition, the tie breaking rule ensures that player i loses with positive probability by choosing s, so he does not win the highest prize with probability 1. In contrast, if player j increases his performance slightly above s, his cost is almost the same but his expected winnings would have a discontinuous increase. This is because he no longer needs to share any prize with player i. This is a deviation for player j, which is a contradiction.

We prove Lemma 2 in two steps. First, suppose two players have an atom at performance level s in the equilibrium, then the above claim implies that both of them must lose with certainty by choosing s. Second, suppose only player i has an atom at s, and suppose he wins a prize with positive probability. On the one hand, if all other players have no best response in $(s-\varepsilon, s)$ for some $\varepsilon > 0$, player i would benefit from lowering the atom to $s-\varepsilon$. This is a contradiction. On the other hand, suppose another player j has a sequence of best responses converging to s from below. Compared to such a best response close to s, performance slightly above s imposes an almost identical cost on player j, but the resulting expected winnings would have a discontinuous increase because of player i's atom at s. This is also a contradiction. In sum, player i loses with certainty by choosing performance level s, which completes the proof. ■

Proof of Lemma 7. We first show that the lowest performance in the supports of the mixed strategies must be zero. To see why, given any equilibrium, let $\underline{s}$ be the lowest performance in the union of all strategies' supports in the equilibrium. Then, at least one player's strategy has $\underline{s}$ as the lower bound of its support. If $\underline{s} > 0$, this player wins the lowest prize with performance $\underline{s}$. On the other hand, he could also win the same prize with performance 0, which incurs a lower cost. This is a contradiction. As a result, we must have $\underline{s} = 0$, so the payoff of this player equals the lowest prize.

Next, we show that that all players have the same payoff. To see this, if the prizes are identical, everyone receives the same prize, hence the lemma is true. Suppose the prizes are not identical. In addition, suppose that in an equilibrium, player i's payoff is lower than player j's. Let $\bar{s}_j$ be the highest performance in the support of player j's

[14] This result is referred to as the Tie Lemma by Siegel (2009).

mixed strategy and $\bar{s}_i$ be the counterpart for i. Lemma 3 implies that player i's expected winnings with performance $\bar{s}_j$ is no lower than player j's. In addition, notice that both players have the same marginal cost, so player i's payoff at $\bar{s}_j$ is no lower than that of j at $\bar{s}_j$. This contradicts the assumption that player j's payoff is higher than i's. Thus, all the players have the same payoff, and denote it as u_t.

Recall that $\underline{s} = 0$, so at least one player, say i, has a equilibrium strategy G_i^m with $\underline{s} = 0$ being the lower bound of its support. Then, given other players equilibrium strategies $\mathbf{G}_{-i}^m$, player i's payoff from choosing $\underline{s} = 0$ is $u_t = \underline{v}^m + \mu_t \sum_{j \neq i} \mathbb{E}[s_j]$, where $\mathbb{E}[s_j]$ is player j's expected performance given his equilibrium strategy G_j^m. By the definition of player i's payoff, we have $\mathbb{E}[W(\mathbf{G}_{-i}^m(s_i), \mathbf{v}^m)] + \mu_t \sum_{j \neq i} \mathbb{E}[s_j] - c_t \mathbb{E}[s_i] = u_t$, where $\mathbb{E}[W(\mathbf{G}_{-i}^m(s_i), \mathbf{v}^m)]$ is player i's expected winnings in the equilibrium. Substituting $u_t = \underline{v}^m + \mu_t \sum_{j \neq i} \mathbb{E}[s_j]$ into the above equation, we can rewrite it as $\mathbb{E}[W(\mathbf{G}_{-i}^m(s_i), \mathbf{v}^m)] - c_t \mathbb{E}[s_i] = \underline{v}^m$. Because the total value of prizes equals the total winnings, aggregating the above equation across all players implies $V^m - c_t S^m = \underline{v}^m |P^m|$ or $S^m = (V^m - \underline{v}^m |P^m|)/c_t$. ■

Proof of Lemma 8. Given others' equilibrium strategies $\mathbf{G}_{-l}^m$, player l can always deviate to non-performance and obtain a payoff no lower than $\underline{v}^m + \sum_{j \neq l} \mathbb{E}[s_j]$. Therefore, every player's equilibrium payoff satisfies $u_l \geq \underline{v}^m + \mu_l \sum_{j \neq l} \mathbb{E}[s_j]$.

Suppose $u_l = \underline{v}^m + \mu_l \sum_{j \neq l} \mathbb{E}[s_j]$ for all players. Then, given others' equilibrium strategies, if player i chooses performance s_i in the support of his equilibrium strategy, he obtains his equilibrium payoff:

$$W(\mathbf{G}_{-i}^m(s_i), \mathbf{v}^m) - c_i s_i + \mu_i \sum_{j \neq i} \mathbb{E}[s_j] = u_i \tag{8}$$

Suppose players i and j have different costs, with $c_i > c_j$. Then, as in the proof of Lemma 5, if player j chooses $\bar{s}_i$, his expected winnings are no lower than i's, but his cost is lower than i's, so

$$W(\mathbf{G}_{-i}^m(\bar{s}_i), \mathbf{v}^m) - c_i \bar{s}_i < W(\mathbf{G}_{-j}^m(\bar{s}_i), \mathbf{v}^m) - c_j \bar{s}_i \tag{9}$$

Notice that the left hand side equals $u_i - \mu_i \sum_{l \neq i} \mathbb{E}[s_l]$ due to (8). In addition, the right hand side cannot be higher than $u_j - \mu_j \sum_{l \neq j} \mathbb{E}[s_l]$, otherwise player j receives a payoff higher than his equilibrium payoff from choosing $\bar{s}_i$. Hence, (9) implies $u_i - \mu_i \sum_{l \neq i} \mathbb{E}[s_l] < u_j - \mu_j \sum_{l \neq j} \mathbb{E}[s_l]$, which contradicts the assumption $u_l = \underline{v}^m + \mu_l \sum_{j \neq l} \mathbb{E}[s_j]$ for all players l. ■

Proof of Proposition 3. As in the proof of Proposition 1, we first discuss the performance outcomes under mixing, and then show that they are dominated by the outcomes under separating. Suppose the players are mixed. Because of spillovers, (5) no longer holds. Therefore, we have to modify the proof of Proposition 1. Specifically, consider a contest with mixed types and an equilibrium in which restrictions (2) and (3) are satisfied. Equation (8) implies $\mathbb{E}[W(\mathbf{G}^m_{-i}(s_i), \mathbf{v}^m)] - c_i\mathbb{E}[s_i] + \mu_i \sum_{j\neq i} \mathbb{E}[s_j] = u_i$, or

$$\mathbb{E}[s_i] = \{\mathbb{E}[W(\mathbf{G}^m_{-i}(s_i), \mathbf{v}^m)] + \mu_i \sum_{j\neq i} \mathbb{E}[s_j] - u_i\}/c_i \tag{10}$$

According to Lemma 8, we have $\mu_i \sum_{j\neq i} \mathbb{E}[s_j] + \underline{v}^m - u_i \leq 0$. Notice that $\underline{v}^m \geq 0$, so we also have

$$\mu_i \sum_{j\neq i} \mathbb{E}[s_j] - u_i \leq 0 \tag{11}$$

for all i and the inequality is strict for at least one player. Then, (10) and (11) imply $\mathbb{E}[s_i] \leq \mathbb{E}[W(\mathbf{G}^m_{-i}(s_i), \mathbf{v}^m)]/c_i$ for all i, and for at least one player the inequality is strict. Recall that W_t is the total expected winnings of t-type players and S_t their total expected performance, so $S_t \leq W_t/c_t$ for all t, and for at least one type the inequality is strict.

Suppose that the planner separates the players into T contests, so each contest contains all the players of a particular type. In addition, suppose she assigns prizes of total value W_t to the contests with t-type players and all the prizes except the lowest are positive. It is straightforward to verify that requirements (2) and (3) are satisfied. In addition, Lemma 7 implies the total expected performance of the t-type contest is W_t/c_t for all t. Hence, as in the proof of Proposition 1, we can verify that the resulting performance outcome dominates the outcome $(S_1, S_2, ..., S_T)$ under mixing. As a result, the weighted total performance is never maximized if the players are mixed. ■

Proof of Proposition 6. Suppose the partition is $(\{1,2\},\{3,4\})$, which means players 1 and 2 are in one contest, and players 3 and 4 in another. Then, players 1 and 2 compete in a contest for a prize of $v_{12} \in (0,1)$, and players 3 and 4 compete in the other contest for a prize of $1 - v_{12}$. The equilibrium payoffs are $u_1 = v_{12}(1 - c_1/c_2)$ for player 1 and $u_2 = 0$ for player 2. The equilibrium strategies are $G_1(s) = c_2 s/v_{12}$ and $G_2(s) = (u_1 + c_1 s)/v_{12}$. Hence, the expected performance of the two players are

$$\begin{aligned}
\mathbb{E}[s_1] &= \int_0^{v_{12}/c_2} s dG_1(s) = \frac{v_{12}}{2c_2} \\
\mathbb{E}[s_2] &= \int_0^{v_{12}/c_2} s dG_2(s) = \frac{v_{12}c_1}{2c_2^2}
\end{aligned}$$

Similarly, the expected performance for players 3 and 4 are $\mathbb{E}[s_3] = (1 - v_{12})/(2c_4)$ and $\mathbb{E}[s_4] = (1 - v_{12})c_3/(2c_4^2)$. Therefore, the maximum total expected performance given the partition is

$$\Pi_{12,34} = \max_{v_{12}\in[0,1]} \left(\frac{v_{12}}{2} \frac{c_1 + c_2}{c_2^2} + \frac{1 - v_{12}}{2} \frac{c_3 + c_4}{c_4^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2} \right)$$

Similarly, the maximum expected performance with other partitions are

$$\Pi_{13,24} = \max_{v_{13}\in[0,1]} \left(\frac{v_{13}}{2} \frac{c_1 + c_3}{c_3^2} + \frac{1 - v_{13}}{2} \frac{c_2 + c_4}{c_4^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_3}{c_3^2}, \frac{c_2 + c_4}{c_4^2} \right)$$

$$\Pi_{14,23} = \max_{v_{14}\in[0,1]} \left(\frac{v_{14}}{2} \frac{c_1 + c_4}{c_4^2} + \frac{1 - v_{14}}{2} \frac{c_2 + c_3}{c_3^2} \right) = \frac{1}{2} \max \left(\frac{c_1 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right)$$

Hence, the maximum expected performance across all partitions is

$$\begin{aligned}
& \max \left(\Pi_{12,34}, \Pi_{13,24}, \Pi_{14,23} \right) \\
= \ & \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2}, \frac{c_1 + c_3}{c_3^2}, \frac{c_2 + c_4}{c_4^2}, \frac{c_1 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right) \\
= \ & \frac{1}{2} \max \left(\frac{c_1 + c_2}{c_2^2}, \frac{c_3 + c_4}{c_4^2}, \frac{c_2 + c_3}{c_3^2} \right)
\end{aligned}$$

where the second equality comes from $\frac{c_1+c_3}{c_3^2} < \frac{c_2+c_3}{c_3^2}$ and $\frac{c_1+c_4}{c_4^2} < \frac{c_2+c_4}{c_4^2} < \frac{c_3+c_4}{c_4^2}$. Therefore, partition $(\{1,3\},\{2,4\})$ is never optimal. Moreover, if $\frac{c_2+c_3}{c_3^2} > \frac{c_1+c_2}{c_2^2}$ and $\frac{c_2+c_3}{c_3^2} > \frac{c_3+c_4}{c_4^2}$, $\Pi_{14,23} > \Pi_{12,34}$. Otherwise, $\Pi_{14,23} \leq \Pi_{12,34}$. ■

Proposition 7 *Suppose player i_t of t-type has a marginal cost $c_t + \varepsilon_{i_t} > 0$, where ε_{i_t} represents the idiosyncratic shock and is commonly known. Then, for any weights, there exists an $\varepsilon > 0$ such that if $\max_{i_t \in N} |\varepsilon_{i_t}| < \varepsilon$, the total weighted expected performance is maximized only if the players are separated.*

Proof. Let us first show the counterpart of Lemma 4: If all the players in a contest are of the same type, their players' payoffs converge to the value of the lowest prize in any equilibrium as $\max_{i_t \in N} |\varepsilon_{i_t}|$ goes to zero. To see why, notice that the player with the highest marginal cost, say player n, has a payoff that equals the lowest prize. According to Lemma 3, player n can ensure himself expected winnings no less than that of player i's at performance level $\bar{s}_i$, player i's highest performance in the support of his strategy. As a result, if players n and i's costs converge towards each other, player n's payoff cannot be lower than i's in the limit. Since no player's equilibrium payoff can be lower than the lowest prize, the payoffs of player i and n must be the same and equal to the lowest prize in the limit.

Let us now show the counterpart of Lemma 5: If not all the players in a contest have identical types and not all the prizes have identical values, at least one player's payoff is higher than and bounded away from the lowest prize in any equilibrium as $\max_{i_t \in N} |\varepsilon_{i_t}|$

goes to zero. To see this, suppose players i and j have different cost types, then their costs in the limit are also different: $c_i > c_j$. In addition, suppose that they have the same equilibrium payoff in the limit, that is, $u_i = u_j$. According to Lemma 3, player j can ensure himself expected winnings no lower than that of player i's at performance level $\bar{s}_i$. Therefore, player j can ensure himself a payoff higher than i's by choosing $\bar{s}_i$ because j has a lower cost in the limit. This is a contradiction.

Given the counterparts of Lemmas 4 and 5, we can prove Proposition 7 in the exact same way as we prove Proposition 1. ■